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Inflation Without Potential

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Due to the importance of inflation, considerable effort has gone into developing different inflationary scenarios. In most of them inflation is driven by a self-interacting scalar field. Here we discuss an alternative way to implement an inflationary stage provided by noncanonical kinetic terms in the action for the scalar field.

1. INTRODUCTION

The homogeneity and flatness of the observable universe naturally follow from inflation, a stage of accelerated expansion of the universe (see, for instance, ref. 1). Moreover, inflation is the only way to explain the origin of the small density fluctuations which led to the observable structure in the universe. For these and many other reasons inflation has become a cornerstone of modern cosmology.

Due to the importance of inflation, considerable effort has gone into developing different inflationary scenarios. In most of them inflation is driven by a self-interacting scalar field. The most general Lagrangian in this case is

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu} - V(\varphi) \tag{1}$$

There are many different models of this type and all of them rely on some sort of "slow-roll" regime during which the scalar field slowly rolls down its potential (see, however, ref. 2). Because the speed of the field is proportional to the slope of the potential, slow-roll inflationary scenarios work only if the corresponding potential is sufficiently "flat."

Of course, the choice of the Lagrangian cannot be arbitrary. Ultimately, the Lagrangian responsible for inflation should stem from high-energy physics. Some of the first inflationary models appeared, for instance, in the context

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of grand unified theories of gauge interactions, but these scenarios lost their appeal with further developments in the subject. Today many particle physicists consider string theory as a candidate for a truly unified theory of gravitation and gauge interactions. Therefore, it would be natural to look for inflation in this framework. Actually, string theory predicts the existence of a whole set of scalar fields known as moduli. However, it is difficult to implement inflation with such scalar fields because they remain massless to all orders in perturbation theory and, even if one includes nonperturbative effects, the nonperturbative potentials are not flat enough [4]. There is nevertheless an alternative way to implement an inflationary stage here provided by noncanonical kinetic terms in the action for the scalar field. Actually, the low-energy effective action of string theory contains such terms for the moduli fields. In particular, its scalar sector takes generically the form

$$\mathcal{L}_{\text{eff}} = -B_g(\phi)R - B_\phi(\phi)(\nabla\phi)^2 + \alpha' B_\phi^{(1)}(\phi)(\nabla\phi)^4 + \cdots \tag{2}$$

As we are going to show, terms like these may lead to a stage of inflation, k-inflation [5]. Let us also point out that even in a nonstringy context k-inflation may be attractive; it provides an in principle totally different way of implement inflation, keeping at the same time its main virtues.

2. k-INFLATION

Consider an effective action containing nonstandard kinetic terms. After conformal transformation if necessary, such an action can be always written as

$$S = \int d^4x \sqrt{-g}\left[-\frac{R}{16\pi G} + p(\varphi, X)\right] \tag{3}$$

The variable X is an abbreviation for the "kinetic term"

$$X = \tfrac{1}{2}(\nabla\varphi)^2 \tag{4}$$

and p is a general function of φ and X. Because we want to address whether inflation without potential is possible, we assume that p can be expanded for small X as

$$p(\varphi, X) = K(\varphi)X + L(\varphi)X^2 + \cdots \tag{5}$$

A scalar field described by (3) mimics the behavior of a perfect fluid with energy-momentum tensor

$$T^\mu_\nu = (\varepsilon + p)u^\mu u_\nu - p\delta^\mu_\nu \tag{6}$$

where the Lagrangian $p(\varphi, X)$ itself plays the role of the pressure, the energy density is given in terms of p by

$$\varepsilon = \varepsilon(\varphi, X) = 2Xp_{,X} - p \tag{7}$$

and the four-velocity is

$$u_\mu = \frac{\nabla_\mu \varphi}{\sqrt{2X}} \tag{8}$$

The equation of motion of the field can be expressed in terms of the ε and p defined above as $\dot{\varepsilon} = -3H(\varepsilon + p)$. We consider only a flat universe and adopt units such that $8\pi G/3 = 1$. Then, energy density and Hubble constant H are related by the Friedmann equation $H^2 = \varepsilon$. In an expanding universe both equations of motion can be combined into the single "master" equation

$$\dot{\varepsilon} = -3\sqrt{\varepsilon}(\varepsilon + p) \tag{9}$$

2.1. Basic Idea of k-Inflation

To get an idea of how nonstandard kinetic terms may support inflation, consider first for simplicity a φ-independent Lagrangian, that is,

$$p = p(X) \tag{10}$$

The energy density in this case is also a function of X alone,

$$\varepsilon = \varepsilon(X) = 2Xp_{,X} - p \tag{11}$$

so that one can draw a parametric plot in the (ε, p) plane which describes the equation of state $p(\varepsilon)$ of the field. Such a plot, for generic p, is shown in Fig. 1.

The crucial observation is that if there is a point X_0 where $p_{,X}(X_0) = 0$ (such points lie on the line $p = -\varepsilon$ in the plot), then $X = X_0 = \text{const}$ is a de Sitter inflationary solution of the master equation (9). Moreover, as can be read out from Fig. 1, by taking into account Eq. (9), these points correspond to attractors of the cosmological evolution of the field.

There are, however, two major drawbacks of a stage of pure de Sitter inflation: There is no exit from inflation and the cosmological perturbations are ill defined (the propagation speed of the perturbations is zero). These drawbacks can be easily avoided by relaxing one of our assumptions, namely, the φ independence of p.

2.2. Slow-Roll k-Inflation

Consider now a φ-dependent p, but in order not to spoil the inflationary behavior we have discussed above, assume this φ dependence to be "mild" (later we shall clearly define what we mean). To simplify the consideration we restrict ourselves to the Lagrangian

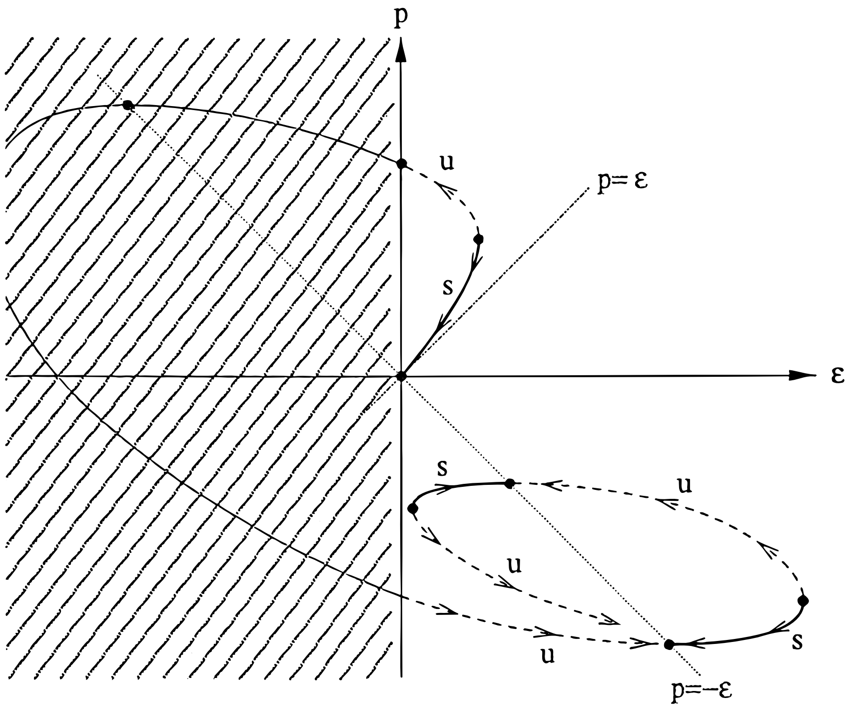

Fig. 1. Equation of state for a rather general kinetic Lagrangian $p(X)$. The evolution for expanding, flat cosmologies proceeds along the indicated arrows. The shaded region ($\varepsilon < 0$) is excluded. Except for the origin and the point above it on the vertical axis, the attractors of the evolution are inflationary fixed points with $p = -\varepsilon$. Along the dashed stretches labeled by u, solutions are absolutely unstable ($c_s^2 < 0$; see Section 3).

$$p = -K(\varphi)X + X^2 \tag{12}$$

although the analysis is applicable to more general Lagrangians. If K does not depend on φ, a point X_0 where

$$p_{,X}(X_0) = 0 \tag{13}$$

would be an inflationary solution of the equations of motion. For a φ-dependent p we can only expect this point to be an approximate lowest order solution of the equations of motion. Note that the solution of (13) for a Lagrangian (12) is

$$X_0 = \tfrac{1}{2}K(\varphi) \tag{14}$$

which of course is not constant, since K depends on φ, and φ changes in time according to

$$\dot{\varphi} = \pm\sqrt{K} \tag{15}$$

Consequently, the energy density also changes with time,

$$\varepsilon_0 \equiv \varepsilon(X_0) = \tfrac{1}{4}K^2(\varphi) \tag{16}$$

In spite of this fact, if the relative change of the energy density during a Hubble time is small, then (14) is a good approximate solution of the equation of motion (9). Using Eq. (15) and the lowest order solution (14), we can express this criterion as

$$\frac{(1/H)\dot{\varepsilon}}{\varepsilon} = -3\,\frac{\varepsilon + p}{\varepsilon} \approx \frac{K'}{K^{3/2}} \ll 1 \tag{17}$$

The last requirement is the equivalent of the "slow-roll condition" on the potential in the usual inflationary scenarios. However, condition (17) is satisfied by a much wider class of functions, such as the following:

1. $K \propto \varphi^{\alpha}$ as $\varphi \to \infty$ for $\alpha > -2$, or as $\varphi \to 0$ for $\alpha < -2$.
2. $K \to$ limit with $K' \to 0$ as $\varphi \to \infty$.
3. $K \propto e^{\varphi}$, $K \propto e^{e^{\varphi}}$, . . . , as $\varphi \to \infty$.

From (17) it also follows, as in the usual models, that during slow-roll k-inflation the universe is close to (but not in) a de Sitter stage. For the functions listed above, condition (17) is satisfied only for certain values of the field. Hence, again in close analogy to the usual models, when during its cosmological evolution the field reaches a region where (17) is violated, inflation naturally ends. A phase diagram showing the whole evolution of the field is shown in Fig. 2.

2.3. Power-Law k-Inflation

The function $K \propto 1/\varphi^2$ does not belong to the class of functions (17). For this K it is possible, however, to find an exact solution of the equations of motion. In fact, one can find solutions for the far more general set of functions

$$p(\varphi, X) = \frac{g(X)}{\varphi^2} \tag{18}$$

Note that the Lagrangian (12) can be cast in this form after a field redefinition. These kinds of Lagrangians also appear for instance from a tree-level low-energy effective string action containing nonstandard kinetic terms by the field redefinition $\varphi \propto e^{-\phi/2}$.

It is easy to verify that if X_0 satisfies the equation

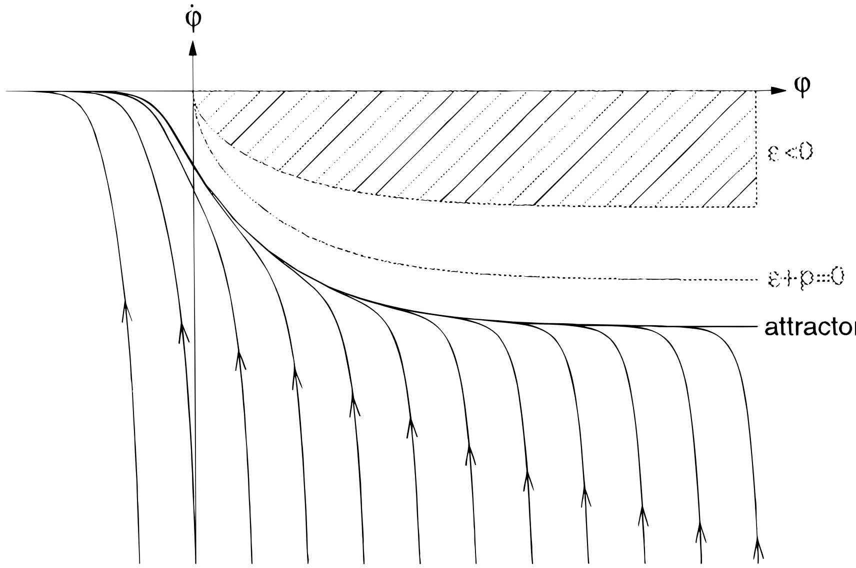

Fig. 2. Shematic phase diagram of "slow-roll" k-inflation. Trajectories approach the inflationary attractor, which does not coincide with the de Sitter line $\varepsilon + p = 0$. Around the point where the slow-roll condition is violated, the solutions leave the inflationary stage and then smoothly approach the vacuum $\dot{\varphi} = 0$.

$$\left(\frac{9}{4}\, g_{,X}^2 - g_{,X} + \frac{g}{2X}\right)_{X_0} = 0 \tag{19}$$

and $g_{,X}(X_0) > 0$, then $X = X_0 = \text{const}$ is a solution of the equation of motion (9) such that

$$a \propto t^{\beta}, \qquad \text{where} \quad \beta = \tfrac{3}{2} g_{,X}(X_0) \tag{20}$$

For $\beta > 1$ the last solution describes the so-called power-law inflation. During power-law inflation, the expansion of the universe is indeed accelerated, $\ddot{a} > 0$, and the energy density gradually decreases as $a^{-2/\beta}$. Notice that in the limit $\beta \to \infty$ one recovers de Sitter inflation. An exit from such power-law inflation can be easily arranged if the $1/\varphi^2$ factor changes its φ dependence in some range of values of φ.

2.4. Pole-Like k-Inflation

The same class of Lagrangians (18) also allows a quite different type of inflation. Namely, if X_0 satisfies (19) and $g(X_0) < 0$, then $X = X_0 = \text{const}$ is a solution of the equation of motion (9) such that

$$a \propto (-t)^{\beta}, \qquad \text{where} \quad \beta = \tfrac{3}{2} g_{,X}(X_0) < 0 \tag{21}$$

Time runs in this case from $-\infty$ to 0, and solution (21) describes pole-like inflation. During pole-like inflation, the energy density and the Hubble constant increase, and the universe becomes singular in a finite time. In contrast to power-law k-inflation, it seems that it is impossible to exit such an inflationary stage.

During power-law and pole-like inflation, the behavior of the scalar field is very different. In both cases the kinetic term $X = X_0$ is constant, so the energy density is proportional to $1/\varphi^2$. However, during power-law inflation the energy density decreases, and therefore the scalar field φ drifts away from the singular point $\varphi = 0$. On the other side, the energy density during pole-like inflation increases and in this case the field φ approaches and reaches the singular point $\varphi = 0$ in a finite time.

3. PERTURBATIONS

Initial inhomogeneities present in the universe before inflation are stretched by the accelerated expansion of the universe and soon become irrelevant in presently observable scales. On the other hand, the new inhomogeneities being generated during inflation from inevitable quantum fluctuations finally become responsible for the observable structure in the universe. Therefore, it is important to analyze the behavior of perturbations during k-inflation [6].

Let us consider small inhomogeneities of the scalar field, that is,

$$\varphi(t, \vec{x}) = \varphi_0(t) + \delta\varphi(t, \vec{x}) \tag{22}$$

When dealing with perturbations in general relativity one should bear in mind that because the energy density and pressure of the constituents of the universe determine its geometry, perturbations in the metric also should be included. The metric of a universe with small inhomogeneities can be written in conformal Newtonian gauge as

$$ds^2 = (1 + \Phi)\, dt^2 - (1 - 2\Phi) a^2\, d\vec{x}^2 \tag{23}$$

The perturbations Φ and $\delta\varphi$ satisfy the linearized Einstein equations, which read

$$\frac{1}{a^2}\, \Delta\Phi - 3H\dot{\Phi} - 3H^2\Phi = \frac{3}{2}\, \delta T^0_0 \tag{24}$$

$$\dot{\Phi} + H\Phi = \frac{3}{2}\, \delta T^0_i \tag{25}$$

and where the perturbations of the energy-momentum tensor are given by

$$\delta T_0^0 = \frac{\epsilon + p}{c_s^2}\left[\left(\frac{\delta\varphi}{\dot{\varphi}}\right)^{\cdot}\right] - 3H\,\frac{\delta\varphi}{\dot{\varphi}} \tag{26}$$

$$\delta T_i^0 = (\epsilon + p)\,\frac{\delta\varphi}{\dot{\varphi}} \tag{27}$$

We have introduced here the squared "speed of sound" defined by

$$c_s^2 = p_{,X}/\varepsilon_{,X} \tag{28}$$

which, as we will see, describes the local speed of propagation of the perturbations. Observe that the ratio p/ε which defines the expansion rate of the universe contains only up to the first X-derivative of p. On the other hand, the speed of sound contains a second X-derivative of p. Thus, p/ε and c_s^2 can be independently arranged to take *a priori* given values by an appropriate choice of the Lagrangian. In contrast, in the usual inflationary models where $p = X - V(\varphi)$ the speed of sound is always equal to one.

The coupled system of differential equations (24), (25) can be simplified by introducing the two new variables ζ and ξ defined via

$$\Phi = \frac{3}{2}\frac{H}{a}\,\xi \qquad \text{and} \qquad \frac{\delta\varphi}{\dot{\varphi}} = \frac{\zeta}{H} - \frac{3}{2a}\,\xi \tag{29}$$

In terms of these new variables, Eqs. (24), (25) reduce to

$$\dot{\xi} = \frac{a(\varepsilon + p)}{H^2}\,\zeta \qquad \text{and} \qquad \dot{\zeta} = \frac{c_s^2 H^2}{a^3(\varepsilon + p)}\,\Delta\xi \tag{30}$$

The variable ζ is useful because its value at the time of recombination is equal, up to a factor of order one, to the amplitude of the temperature fluctuations in the cosmic microwave background radiation on large angular scales,

$$\frac{\delta T}{T} = O(1)\cdot\zeta\big|_{\text{rec}} \tag{31}$$

Furthermore, the appropriate variable for canonical quantization v is proportional to ζ,

$$v = z\zeta, \qquad \text{where} \quad z = \frac{a}{c_s}\left(1 + \frac{p}{\varepsilon}\right)^{1/2} \tag{32}$$

The equation of motion of v can be deduced from Eqs. (30). It reads

$$v'' - c_s^2\Delta v - \frac{z''}{z}\,v = 0 \tag{33}$$

where a prime means derivative with respect to conformal time $\eta = \int dt/a$.

Notice that c_s^2 plays indeed the role of a propagation speed. Strictly, the last interpretation is right only if c_s^2 is positive. If not, perturbations grow with time instead of oscillating, and the background solutions become absolutely unstable. Thus, stability requires c_s^2 to be positive.

During slow-roll k-inflation c_s and $1 + p/\varepsilon$ change much more slowly than the scale factor. Thus, from (33) we have $z''/z \approx 2H^2a^2$, where we have used (17). If we decompose the variable v into Fourier modes v_k, the equation satisfied by each mode is decoupled from the others. Each Fourier mode v_k describes the field fluctuations in a particular comoving length scale $\lambda \approx 1/k$. For modes "inside the sound horizon" ($a/k \ll c_s/H$), solutions are oscillatory. In this short-wave limit, the solution which describes the vacuum, the state of minimal fluctuations of the field, corresponds to

$$v_k \approx \frac{e^{-ikc_s\eta}}{(2c_sk)^{1/2}} \tag{34}$$

Notice that because of the fast change in the scale factor, modes which are initially inside the sound horizon soon get "out of the sound horizon" ($a/k \gg c_s/H$), so that we can use the short-wave vacuum solution as an "initial condition" to compute the amplitude of the long-wave modes. For these modes, nondecaying solutions are proportional to z. The constant of proportionality is found by matching this solution with the vacuum one at "sound horizon crossing" ($c_sk = aH$). One finds then that the quantity which characterizes the squared amplitude of the temperature fluctuations, the power spectrum $\mathcal{P}_k^\zeta$, is given on large scales by

$$\mathcal{P}_k^\zeta \equiv \frac{1}{2\pi^2}\,|\zeta_k|^2k^3 = \frac{1}{4\pi^2}\,\frac{\varepsilon}{\varepsilon_{\rm Pl}}\,\frac{1}{c_s(1 + p/\varepsilon)}\bigg|_{c_sk\approx aH} \tag{35}$$

where, as indicated, the appropriate quantities should be evaluated at the moment of sound horizon crossing. Observe that the speed of sound is small during slow-roll k-inflation (at least in the model in consideration), whereas in power-law inflation there is no *a priori* restriction on its value.

An important consequence of the previous formula is the fact that the ratio of the power spectrum of gravitational waves $\mathcal{P}^h$ to the power spectrum of the scalar variable ζ is given by

$$\frac{\mathcal{P}^h}{\mathcal{P}^\zeta} = -8c_sn_T \tag{36}$$

where the tensor spectral index n_T describes the slope of the spectrum of gravitational waves. Hence, by measuring n_T, $\mathcal{P}^h$, and $\mathcal{P}^\zeta$, one can determine in principle c_s. In standard inflation c_s is always one, but in k-inflation this restriction does not hold. Thus, k-inflation is phenomenologically distinguishable from usual inflation!

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